

ENERGY IDENTITY FOR APPROXIMATE HARMONIC MAPS FROM SURFACE TO GENERAL TARGETS

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ABSTRACT. Let u_n be a sequence of mappings from a closed Riemannian surface M to a general Riemannian manifold N . If u_n satisfies

$$\sup_n (\|\nabla u_n\|_{L^2(M)} + \|\tau(u_n)\|_{L^p(M)}) \leq \Lambda \quad \text{for some } p > 1,$$

where $\tau(u_n)$ is the tension field of u_n , then there hold the so called energy identity and neckless property during blowing up. This result is sharp by Parker's example, where the tension fields of the mappings from Riemannian surface are bounded in $L^1(M)$ but the energy identity fails.

1. INTRODUCTION

Let (M, g) be a closed Riemannian manifold and (N, h) be a Riemannian manifold without boundary. Let u be a mapping from M to N in $W^{1,2}(M, N)$. We define the Dirichlet energy of u as follows

$$E(u) = \int_M e(u) dV,$$

where dV is the volume element of (M, g) , and $e(u)$ is the density of u

$$e(u) = \frac{1}{2} |du|^2 = \text{Trace}_g u^* h,$$

where $u^* h$ is the pull-back of the metric tensor h .

A map $u \in C^1(M, N)$ is called harmonic if it is a critical point of the energy E . By the Nash embedding theorem, (N, h) can be isometrically embedded into a Euclidean space \mathbb{R}^k for some positive integer k with the metric induced from the Euclidean metric. Hence, a map $u \in C^1(M, N)$ can be viewed as a map of $C^1(M, \mathbb{R}^k)$ whose image lies in N . Then we can obtain the Euler-Lagrange equation

$$(1.1) \quad \Delta u - A(u)(du, du) = 0, \quad \text{or} \quad P(u)\Delta u = 0,$$

where $A(u)(du, du)$ is the second fundamental form of N in \mathbb{R}^k . Let $P(y) : \mathbb{R}^k \rightarrow T_y N$ be the orthogonal projection map. The tension field $\tau(u)$ is defined by

$$(1.2) \quad \tau(u) \stackrel{\text{def}}{=} \Delta u - A(u)(du, du) = P(u)\Delta u.$$

Then u is harmonic if and only if $\tau(u) = 0$. We refer to [7] for the systematic study on the harmonic maps.

The harmonic maps are of special interest when M is a Riemannian surface, because the Dirichlet energy is conformally invariant in two dimensions. It is an important question to understand the limiting behavior of sequences of harmonic maps. Let u_n be a sequence of mappings from Riemannian surface M to N with bounded energy. It is clear that u_n converges weakly to u in $W^{1,2}(M, N)$ for some $u \in W^{1,2}(M, N)$. In general, it may not

converge strongly in $W^{1,2}(M, N)$ due to the concentration of the energy at finitely many points [13]. Thus, it is natural to ask (1) whether the lost energy is exactly the sum of energies of some harmonic spheres(bubbles), which are defined as harmonic maps from \mathbb{S}^2 to N ; (2) whether attaching all possible bubbles to the weak limit gives uniform convergence. The first one is so called the energy identity, and the second one is called the bubble tree convergence.

When $\tau(u_n) = 0$, Jost and Parker [10] independently proved the energy identity and neckless property during blowing up. When $\tau(u_n)$ is bounded in $L^2(M)$, the energy identity was proved by Qing [11] for the sphere, by Ding and Tian [1] and Wang [16] for general target manifold. Qing and Tian [12] also proved neckless property during blowing up. One can refer to [14, 15, 5] for the related results of the heat flow of harmonic maps. Notice that $L^2(M)$ space for the tension field is not conformally invariant. So, a natural substitution of $L^2(M)$ space seems $L^1(M)$ space. However, Parker [10] construct a sequence of mappings from Riemannian surface, in which the tension fields are bounded in $L^1(M)$ but the energy identity fails. This motivates the following important question:

Q: whether the energy identity or the neckless property holds for a general target manifold when the tension field is bounded in $L^p(M)$ for $p > 1$?

This question has been solved by Lin and Wang [6] when the target manifold is the sphere. Li and Zhu [3] also proved the energy identity for the tension fields bounded in $L \ln^+ L$, and constructed a sequence of mappings with tension fields bounded in $L \ln^+ L$ so that there is a positive neck during blowing up. For general target manifolds, partial important progress has been made: Li and Zhu [4] proved the energy identity and the neckless property for $p \geq \frac{6}{5}$; Luo [8] obtained the same result under the following condition

$$\left(\int_{D_r \setminus D_{r/2}} |\tau(u_n)|^2 dx \right)^{\frac{1}{2}} \leq Cr^{-a}$$

for some $a \in (0, 1)$ and any $r \in (0, 1)$. Here we denote by $D(x, r)$ the ball with the center x and the radius r and $D_r = D(0, r)$.

The goal of this paper is to give a positive answer to **Q**. When $\tau(u_n)$ is bounded in $L^p(M)$ for some $p > 1$, the small energy regularity (see Lemma 2.1) implies that u_n converges strongly in $W^{1,2}(M, N)$ outside a finite set of points. For simplicity, we assume that M is the unit disk $D_1 = D(0, 1)$ and 0 is the only one singular point.

Our main result is stated as follows.

Theorem 1.1. *Let $\{u_n\}$ be a sequence of mappings from D_1 to N in $W^{1,2}(D_1, N)$ with tension field $\tau(u_n)$ satisfying*

$$\|u_n\|_{\dot{W}^{1,2}(D_1)} + \|\tau(u_n)\|_{L^p(D_1)} \leq \Lambda$$

for some $p > 1$, and for $0 < \delta < 1$,

$$u_n \rightarrow u \text{ strongly in } W^{1,2}(D_1 \setminus D_\delta, \mathbb{R}^k) \quad \text{as } n \rightarrow \infty.$$

Then there exist a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and some nonnegative integer k_0 such that for any $i = 1, \dots, k_0$, there exist points x_n^i , positive numbers r_n^i and a nonconstant harmonic sphere w_i (a map from $\mathbb{R}^2 \cup \{\infty\} \rightarrow N$), which satisfy

1. for any $i = 1, \dots, k_0$, $x_n^i \rightarrow 0, r_n^i \rightarrow 0$ as $n \rightarrow \infty$.

2.

$$\lim_{n \rightarrow \infty} \left(\frac{r_n^i}{r_n^j} + \frac{r_n^j}{r_n^i} + \frac{|x_n^i - x_n^j|}{r_n^i + r_n^j} \right) = \infty \quad \text{for any } i \neq j.$$

3. w^i is the weak limit or strong limit of $u_n(x_n^i + r_n^i x)$ in $W_{loc}^{1,2}(\mathbb{R}^2, N)$.4. **Energy identity**

$$\lim_{n \rightarrow \infty} E(u_n, D_1) = E(u, D_1) + \sum_{i=1}^{k_0} E(w^i).$$

5. **Neckless property:** The image $u(D_1) \cup \bigcup_{i=1}^{k_0} w^i(\mathbb{R}^2)$ is a connected set.

2. BUBBLE TREE STRUCTURE OF APPROXIMATE HARMONIC MAPS

Let us first recall the following small energy regularity result [1, 3].

Lemma 2.1. *Let u be a mapping from D_1 to N in $W^{1,2}(D_1, N)$ with tension field $\tau(u) \in L^p(D_1)$ for $p > 1$. Then there exists a positive constant ϵ_N depending on the target manifold N such that if $E(u, D_1) \leq \epsilon_N^2$, then*

$$\|u - \bar{u}\|_{W^{2,p}(D_{\frac{1}{2}})} \leq C (\|\nabla u\|_{L^2(D_1)} + \|\tau(u)\|_{L^p(D_1)}),$$

where \bar{u} is the mean value of u on the disk $D_{\frac{1}{2}}$.

Let u_n be a sequence of mapping from D_1 to N satisfying

$$\|u_n\|_{W^{1,2}(D_1)} + \|\tau(u_n)\|_{L^p(D_1)} \leq \Lambda$$

for some $p > 1$. A point $x \in D_1$ is called an energy concentration point (blow-up point) of u_n if for any r so that $D(x, r) \subset D_1$, we have

$$\limsup_{n \rightarrow \infty} E(u_n, D(x, r)) > \epsilon_N^2.$$

Based on Lemma 2.1, using standard blow-up argument as in [11, 1], it can be proved that for fixed sufficiently small $\epsilon \in (0, \epsilon_N)$, there exists $k_0 \geq 0$ so that for any $i = 1, \dots, k_0$, there exist a point x_n^i , a positive number r_n^i , and a nonconstant harmonic sphere w^i satisfying the conclusions 1 – 3 in Theorem 1.1. Moreover, it holds that

1. w^i is the strong limit of $u_n(x_n^i + r_n^i x)$ in $W_{loc}^{1,2}(\mathbb{R}^2 \setminus Z_i, N)$, where Z_i is the set of blow-up points of this scaling sequence, thus Z_i is finite and for $x \in Z_i$,

$$m_i(x) \triangleq \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} E(u_n, D(x_n^i + r_n^i x, \delta r_n^i)) \geq \epsilon_N.$$

For the sake of completeness, we denote $x_n^0 = 0$, $r_n^0 = 1$, $Z_0 = \{0\}$, $w^0 = u$.

2. $\exists f : \{1, \dots, k_0\} \rightarrow \mathbb{Z}$ and $\delta_0, R_0 > 0$ so that

$$0 \leq f(j) < j, \quad \lim_{n \rightarrow \infty} \frac{r_n^j}{r_n^{f(j)}} = 0, \quad \lim_{n \rightarrow \infty} \frac{x_n^j - x_n^{f(j)}}{r_n^{f(j)}} = y_j \in Z_{f(j)},$$

$$\text{and } E(u_n, D(x_n^j, r_n^{f(j)} \delta_0) \setminus D(x_n^j, r_n^j R_0)) \leq \epsilon.$$

3. If $f(i) = f(j)$ and $y_i = y_j$, then $i = j$, $Z_i = \{y_j | f(j) = i\}$.

Let us just present a sketch for the construction of (x_n^1, r_n^1) (see P.118-121 in [9] for similar construction). As $m_0(0) = \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} E(u_n, D_\delta) \geq \epsilon_N$, there exists $\delta > 0$ so that

$$|E(u_n, D_\delta) - m_0(0)| \leq \frac{\epsilon}{4}$$

for n sufficiently large. Let $Q_n(t) \triangleq \sup_{D(z,t) \subseteq D_\delta} E(u_n, D(z,t))$ for $0 \leq t \leq \delta$. Then $Q_n(t)$ is continuous and non-decreasing in t , $Q_n(0) = 0$ and $Q_n(\delta) = E(u_n, D_\delta)$. Therefore, there exists $0 < r_n^1 < \delta$ such that $Q_n(r_n^1) = \max(Q_n(\delta) - \epsilon, \frac{\epsilon}{2})$ and there exists $D(x_n^1, r_n^1) \subseteq D_\delta$ such that $E(u_n, D(x_n^1, r_n^1)) = Q_n(r_n^1)$. Thus, we have $x_n^1 \rightarrow 0$, $r_n^1 \rightarrow 0$ and

$$\begin{aligned} E(u_n, D(x_n^1, \delta) \setminus D(x_n^1, r_n^1)) &\leq E(u_n, D_\delta) - E(u_n, D(x_n^1, r_n^1)) \\ &= Q_n(\delta) - Q_n(r_n^1) \leq \epsilon \end{aligned}$$

for n sufficiently large. Hence, we can take $f(1) = 0$. Moreover, by Lemma 2.1, $u_n(x_n^1 + r_n^1 x)$ has a subsequence, which strongly converges in $W_{loc}^{1,2}(\mathbb{R}^2 \setminus Z_1, N)$ to w^1 , and for $x \in Z_1$,

$$m_1(x) \leq \limsup_{n \rightarrow \infty} Q_n(r_n^1) \leq \max\left(m_0(0) - \frac{3}{4}\epsilon, \frac{\epsilon}{2}\right),$$

which implies that this construction can only happen finite times.

Remark 2.2. In fact, Zhu [18] proved the bubble tree theorem under the weaker condition

$$\|u_n\|_{W^{1,2}(D_1)} + \|\tau(u_n)\|_{L \ln^+ L(D_1)} \leq \Lambda,$$

where $\|f\|_{L \ln^+ L(D_1)} \stackrel{\text{def}}{=} \int_{D_1} |f(x)| \ln(2 + |f(x)|) dx$.

Now, by Property 1 and Lemma 2.1, for fixed $0 < \delta < 1 < R$ and $1 \leq i \leq k_0$, we have

$$\begin{aligned} u_n(x_n^i + r_n^i x) &\rightarrow w^i \text{ strongly in } W^{1,2}(D_R \setminus \cup_{x \in Z_i} D(x, \delta)) \cap C^0(D_R \setminus \cup_{x \in Z_i} D(x, \delta)), \\ u_n &\rightarrow u \text{ strongly in } W^{1,2}(D_1 \setminus D_\delta) \cap C^0(D_{\frac{1}{2}} \setminus D_\delta), \end{aligned}$$

as $n \rightarrow \infty$. Therefore, as $n \rightarrow \infty$,

$$\begin{aligned} E(u_n, D(x_n^i, r_n^i R) \setminus \cup_{x \in Z_i} D(x_n^i + r_n^i x, r_n^i \delta)) &\rightarrow E(w^i, D_R \setminus \cup_{x \in Z_i} D(x, \delta)), \\ E(u_n, D_1 \setminus D_\delta) &\rightarrow E(u, D_1 \setminus D_\delta). \end{aligned}$$

Moreover,

$$\limsup_{n \rightarrow \infty} \text{osc} \left(u_n, D(x_n^i, r_n^{f(i)} \delta) \setminus D(x_n^i, r_n^i R) \right) \geq \text{diam}(w^{f(i)}(\partial D_\delta) \cup w^i(\partial D_R)),$$

and for n sufficiently large,

$$\begin{aligned} &\left| E(u_n, D_1) - E(u_n, D_1 \setminus D_\delta) - \sum_{i=1}^{k_0} E(u_n, D(x_n^i, r_n^i R) \setminus \cup_{x \in Z_i} D(x_n^i + r_n^i x, r_n^i \delta)) \right| \\ &\leq \sum_{i=1}^{k_0} E(u_n, D(x_n^i, r_n^{f(i)} \delta) \setminus D(x_n^i, r_n^i R)). \end{aligned}$$

Thus, the energy identity is equivalent to show that there is no energy on the neck during blow-up process, i.e.,

$$(2.1) \quad \lim_{\delta \rightarrow 0} \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} E(u_n, D(x_n^i, r_n^{f(i)} \delta) \setminus D(x_n^i, r_n^i R)) = 0 \quad \text{for } i = 1, \dots, k_0.$$

While, the neckless property is equivalent to show that there is no oscillation on the neck, i.e.,

$$(2.2) \quad \lim_{\delta \rightarrow 0} \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \text{osc}(u_n, D(x_n^i, r_n^{f(i)} \delta) \setminus D(x_n^i, r_n^i R)) = 0 \quad \text{for } i = 1, \dots, k_0.$$

In order to prove (2.1) and (2.2), our key idea is to show that the Hopf differential of the approximate harmonic map u can be approximated by a holomorphic function, where the error is quantized by the tension field of u . The result is trivial in the case when $\tau(u) = 0$, because the Hopf differential of harmonic map u is holomorphic.

3. THE COULOMB GAUGE FRAME

Consider $\Omega = D_1$ or $D_1 \setminus D_r$ for $0 < r \leq \frac{1}{4}$. Assume that

$$(3.1) \quad E(u, \Omega) \leq \epsilon_0,$$

where $\epsilon_0 > 0$ will be determined later.

Recall that (N, h) can be isometrically embedded into \mathbb{R}^k . Let \overline{N} be a submanifold of \mathbb{R}^{2k} defined by

$$\overline{N} \stackrel{\text{def}}{=} \{(y, y') \in \mathbb{R}^k \times \mathbb{R}^k : y \in N, y' \perp T_y N\}.$$

Then $N = N \times \{0\}$ is a totally geodesic submanifold of \overline{N} . As in [2, 7], we may introduce the Coulomb gauge frame of $u^*T\overline{N}$. Let us present the construction of Coulomb gauge.

The following lemma makes use of Hardy space (for example, see [2, 7]).

Lemma 3.1. *If $\Delta\Phi = \frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2} \frac{\partial g}{\partial x_1}$, then we have*

$$\|d\Phi\|_{L^{2,1}(\mathbb{R}^2)} \leq C \|df\|_{L^2(\mathbb{R}^2)} \|dg\|_{L^2(\mathbb{R}^2)}.$$

Here $L^{p,q}(\mathbb{R}^2)$ is the Lorentz space.

We also need the following extension lemma.

Lemma 3.2. *Let $f \in W^{1,2}(\Omega)$. Then there exists an extension $\bar{f} \in \dot{W}^{1,2}(\mathbb{R}^2)$ of f so that*

$$E(\bar{f}, \mathbb{R}^2) \leq CE(f, \Omega).$$

Here C is a constant independent of r .

Proof. We only consider the case $\Omega = D_1 \setminus D_r$. First of all, we can find $f_j \in \dot{W}^{1,2}(\mathbb{R}^2)$, $j = 1, 2$ such that $E(f_j, \mathbb{R}^2) \leq CE(f_j, D_1 \setminus D_{\frac{1}{2}})$, and $f_1(z) = f(z)$, $f_2(z) = f(2rz)$ in $D_1 \setminus D_{\frac{1}{2}}$. Then we can extend f to $\dot{W}^{1,2}(\mathbb{R}^2)$ by taking $\bar{f}(z) = f_1(z)$ for $|z| > 1$ and $\bar{f}(z) = f_2(z/2r)$ for $|z| < r$. Then we find that

$$\begin{aligned} E(\bar{f}, \mathbb{R}^2) &= E(\bar{f}, D_{2r}) + E(\bar{f}, D_{\frac{1}{2}} \setminus D_{2r}) + E(\bar{f}, \mathbb{R}^2 \setminus D_{\frac{1}{2}}) \\ &= E(f_2, D_1) + E(f, D_{\frac{1}{2}} \setminus D_{2r}) + E(f_1, \mathbb{R}^2 \setminus D_{\frac{1}{2}}) \\ &\leq CE(f_2, D_1 \setminus D_{\frac{1}{2}}) + E(f, D_{\frac{1}{2}} \setminus D_{2r}) + CE(f_1, D_1 \setminus D_{\frac{1}{2}}) \\ &= CE(f, D_{2r} \setminus D_r) + E(f, D_{\frac{1}{2}} \setminus D_{2r}) + CE(f, D_1 \setminus D_{\frac{1}{2}}) \\ &\leq CE(f, D_1 \setminus D_r). \end{aligned}$$

This completes the proof. \square

Let $\mathcal{A}(\Omega)$ be the set of $R = (e_1, \dots, e_k) \in \dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^{2k \times k})$ such that

$$R^T R = I_k, \quad R R^T = \begin{pmatrix} P(u) & \\ & I_k - P(u) \end{pmatrix} \quad \text{in } \Omega.$$

First of all, we show that $\mathcal{A}(\Omega)$ is nonempty and $E(R) = \frac{1}{2} \int_{\mathbb{R}^2} |dR|^2$ attains a minimum in $\mathcal{A}(\Omega)$.

Indeed, let $R_0 = \begin{pmatrix} P(u) \\ I_k - P(u) \end{pmatrix}$ in Ω and extend R_0 to \mathbb{R}^2 so that $\int_{\mathbb{R}^2} |dR_0|^2 \leq C \int_{\Omega} |dR_0|^2$. Then $R_0 \in \mathcal{A}(\Omega)$ and $E(R_0) \leq CE(u, \Omega)$. Let $R_n \in \mathcal{A}(\Omega)$ so that

$$\lim_{n \rightarrow \infty} E(R_n) = E_0 = \inf_{R \in \mathcal{A}(\Omega)} E(R).$$

Then there exists a subsequence of $\{R_n\}$ (still denoted by $\{R_n\}$) and $R \in \dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^{2k \times k})$ such that $R_n \rightarrow R$ strongly in $L^2(\Omega)$ as $n \rightarrow \infty$, and $dR_n \rightarrow dR$ weakly in $L^2(\mathbb{R}^2)$ as $n \rightarrow \infty$. Then $R_n^T R_n \rightarrow R^T R$ and $R_n R_n^T \rightarrow R R^T$ strongly in $L^1(\Omega)$ as $n \rightarrow \infty$, and $E(R) \leq E_0$. Therefore, $R \in \mathcal{A}(\Omega)$ and $E(R) = E_0 \leq E(R_0) \leq CE(u, \Omega)$.

Now, for $\psi \in C_0^\infty(\mathbb{R}^2, so(k))$, we have $R \exp t\psi \in \mathcal{A}(\Omega)$ for $t \in \mathbb{R}$. Therefore, $E(R \exp t\psi) \geq E_0 = E(R)$ and

$$\begin{aligned} 0 &= \left. \frac{d}{dt} \right|_{t=0} E(R \exp t\psi) = \int_{\mathbb{R}^2} \left\langle dR, \left. \frac{d}{dt} \right|_{t=0} d(R \exp t\psi) \right\rangle = \int_{\mathbb{R}^2} \langle dR, d(R\psi) \rangle \\ &= \int_{\mathbb{R}^2} (\langle dR, R(d\psi) \rangle + \langle dR, (dR)\psi \rangle) = \int_{\mathbb{R}^2} \langle dR, R(d\psi) \rangle = \int_{\mathbb{R}^2} \langle R^T dR, d\psi \rangle, \end{aligned}$$

as $\langle dR, (dR)\psi \rangle = 0$. For $\psi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^{k \times k})$, we have $\psi^T - \psi \in C_0^\infty(\mathbb{R}^2, so(k))$. So,

$$\begin{aligned} 0 &= \int_{\mathbb{R}^2} \langle R^T dR, d(\psi^T - \psi) \rangle = \int_{\mathbb{R}^2} \langle (R^T dR)^T - R^T dR, d\psi \rangle \\ &= \int_{\mathbb{R}^2} \langle (dR^T)R - R^T dR, d\psi \rangle. \end{aligned}$$

Therefore, $d^*((dR^T)R - R^T dR) = 0$ and there exists $\Phi \in \dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^{k \times k})$ so that

$$\frac{1}{2}((dR^T)R - R^T dR) = \frac{\partial \Phi}{\partial x_2} dx_1 - \frac{\partial \Phi}{\partial x_1} dx_2.$$

Noticing that

$$\triangle \Phi = \frac{\partial R^T}{\partial x_1} \frac{\partial R}{\partial x_2} - \frac{\partial R^T}{\partial x_2} \frac{\partial R}{\partial x_1},$$

we infer from Lemma 3.1 that

$$\|d\Phi\|_{L^{2,1}(\mathbb{R}^2)} \leq C \|dR^T\|_{L^2(\mathbb{R}^2)} \|dR\|_{L^2(\mathbb{R}^2)} = C \|dR\|_{L^2(\mathbb{R}^2)}^2 \leq CE(u, \Omega).$$

Thanks to $0 = dI_k = d(R^T R) = (dR^T)R + R^T dR$ in Ω , we have

$$\begin{aligned} (dR^T)R &= \frac{\partial \Phi}{\partial x_2} dx_1 - \frac{\partial \Phi}{\partial x_1} dx_2, \quad d^*((dR^T)R) = 0 \text{ in } \Omega, \\ \|(dR^T)R\|_{L^{2,1}(\Omega)} &= \|d\Phi\|_{L^{2,1}(\Omega)} \leq CE(u, \Omega). \end{aligned}$$

We introduce

$$A = R^T \begin{pmatrix} \frac{\partial u}{\partial z} \\ 0 \end{pmatrix}, \quad \tau^1 = \frac{1}{4} R^T \begin{pmatrix} \tau \\ 0 \end{pmatrix}, \quad w = \frac{\partial R^T}{\partial \bar{z}} R \in so(k) \otimes \mathbb{C} \quad \text{for } z \in \Omega.$$

Then the system of (1.1) is equivalent to

$$(3.2) \quad \frac{\partial A}{\partial \bar{z}} = wA + \tau^1,$$

where w satisfies

$$\|w\|_{L^{2,1}(\mathbb{R}^2)} \leq C\epsilon_0.$$

Let us introduce a linear operator $T : L^\infty(\mathbb{C}, \mathbb{C}^{k \times k}) \rightarrow L^\infty(\mathbb{C}, \mathbb{C}^{k \times k})$ defined by

$$T(B)(z) = \left(\left(\frac{1}{\pi z} \right) * (wB) \right)(z) = \int_{\mathbb{C}} \frac{w(\zeta)B(\zeta)}{\pi(z - \zeta)} d\zeta.$$

Thanks to $\frac{1}{\pi z} \in L^{2,\infty}(\mathbb{C})$ and $w \in L^{2,1}(\mathbb{C})$, we deduce that T maps $L^\infty(\mathbb{C}, \mathbb{C}^{k \times k})$ continuously to itself with the bound

$$(3.3) \quad \|T\| \leq C\|w\|_{L^{2,1}(\mathbb{R}^2)} \leq C\|\nabla u\|_{L^2(\Omega)}^2.$$

Moreover, it holds that

$$(3.4) \quad \frac{\partial}{\partial \bar{z}} T(B) = wB.$$

Lemma 3.3. *If $\|T\| \leq \frac{1}{3}$, then there exists a nonsingular matrix $B \in L^\infty(\mathbb{C}, \mathbb{C}^{k \times k})$ so that*

$$B - T(B) = I_k, \quad B^T B = I_k.$$

Here I_k is the $k \times k$ identity matrix.

Proof. Due to $\|T\| \leq \frac{1}{3}$, we get by the fixed point theorem that

$$B - T(B) = I_k$$

has a unique solution $B \in L^\infty(\mathbb{C}, \mathbb{C}^{k \times k})$ with

$$\|B - I_k\|_{L^\infty(\mathbb{C})} \leq \frac{3}{2}\|T\| \leq \frac{1}{2}.$$

Recall that $w^T = -w$ and $\frac{\partial}{\partial \bar{z}} B = \frac{\partial}{\partial \bar{z}} T(B) = wB$. Thus, we get

$$\frac{\partial}{\partial \bar{z}} B^T = B^T w^T,$$

which implies that

$$\frac{\partial}{\partial \bar{z}} (B^T B) = B^T (w^T + w) B = 0.$$

That means that $B^T B$ is holomorphic. On the other hand, $\lim_{z \rightarrow \infty} B = I_k$. Then

$$B^T B = I_k.$$

The proof is finished. □

4. HOLOMORPHIC APPROXIMATION OF HOPF DIFFERENTIAL

Throughout this section, let us assume that u is a mapping from D_1 to N in $W^{1,2}(D_1, N)$ with the tension field $\tau(u) \in L^p(D_1)$ for some $p \in (1, 2)$. We denote by $h(z)$ the Hopf differential of u , i.e.,

$$h(z) \stackrel{\text{def}}{=} \left\langle \frac{\partial u}{\partial z}, \frac{\partial u}{\partial z} \right\rangle = \sum_{j=1}^k \frac{\partial u^j}{\partial z} \frac{\partial u^j}{\partial z}.$$

It was well-known that if u is harmonic, then $h(z)$ is holomorphic. In this section, we will show that in general case, $h(z)$ can be approximated by a holomorphic function, where the error is quantized by $\|\tau(u)\|_{L^p(D_1)}$. This result may be independent of interest.

Proposition 4.1. *Assume that $E(u, D_1) \leq m\epsilon$ for some $m \in \mathbb{N}$, where ϵ is the minimum of ϵ_0 given by Lemma 4.2 and Lemma 4.3. Then there exist C_m and a holomorphic function h_0 in $D_{1/4}$ such that*

$$\|h - h_0\|_{L^1(D_{1/4})} \leq C_m \|\tau(u)\|_{L^p(D_1)}^{3^{1-m}}.$$

The proof of Proposition 4.1 is based on the following lemmas.

Lemma 4.2. *There exists $\epsilon_0 > 0$ such that if $E(u, \Omega) \leq \epsilon_0$, $\Omega = D_1$ or $D_1 \setminus D_r$ for some $0 < r \leq \frac{1}{4}$, then there exists a holomorphic function h_0 in Ω so that*

$$\|h - h_0\|_{L^1(\Omega)} \leq C \|\tau(u)\|_{L^p(\Omega)}.$$

Proof. Thanks to $E(u, \Omega) \leq \epsilon_0$, by (3.3) we can take ϵ_0 small enough so that $\|T\| \leq C\epsilon_0^2 \leq \frac{1}{3}$. Then by Lemma 3.3, there exists a nonsingular matrix $B \in L^\infty(\mathbb{C}, \mathbb{C}^{k \times k})$ so that

$$B - T(B) = I_k, \quad B^T B = I_k, \quad \|B - I_k\|_{L^\infty(\mathbb{C})} \leq \frac{3}{2} \|T\| \leq C\epsilon_0^2.$$

Let $A = BG$. Then $\frac{\partial}{\partial \bar{z}} G = B^{-1} \tau^1$. We write $G = G_1 + G_2$ with $G_2 = \frac{1}{\pi z} * (B^{-1} \tau^1)$. Then it holds that

$$\frac{\partial G_2}{\partial \bar{z}} = B^{-1} \tau^1, \quad \frac{\partial G_1}{\partial \bar{z}} = 0 \quad \text{in } \Omega.$$

Moreover, it holds that

$$\|G_2\|_{L^{\frac{2p}{2-p}}(\mathbb{R}^2)} \leq C \|\tau^1\|_{L^p(\mathbb{R}^2)} \leq C \|\tau(u)\|_{L^p(\Omega)}.$$

Let $h_1(z) = G_1(z)^T G_1(z)$, which is holomorphic in Ω . Notice that

$$h(z) = A^T A = G^T B^T B G = G^T G.$$

Hence, by $G = B^{-1}A$ and $\|A\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}$, we deduce that

$$\begin{aligned} \|h - h_1\|_{L^p(\Omega)} &= \|G^T G - G_1^T G_1\|_{L^p(\Omega)} \\ &\leq \|G_2\|_{L^{\frac{2p}{2-p}}(\Omega)} (\|G\|_{L^2(\Omega)} + 2\|G_2\|_{L^2(\Omega)}) \\ &\leq C \|\tau(u)\|_{L^p(\Omega)} (\|\nabla u\|_{L^2(\Omega)} + \|\tau(u)\|_{L^p(\Omega)}) \\ &\leq C \|\tau(u)\|_{L^p(\Omega)} (1 + \|\tau(u)\|_{L^p(\Omega)}). \end{aligned}$$

On the other hand, we have

$$\|h\|_{L^1(\Omega)} \leq \left\| \frac{\partial u}{\partial \bar{z}} \right\|_{L^2(\Omega)}^2 \leq E(u, \Omega) \leq 1.$$

This concludes that

$$\begin{aligned} \min\{\|h - h_1\|_{L^1(\Omega)}, \|h\|_{L^1(\Omega)}\} &\leq \min\{C\|\tau(u)\|_{L^p(\Omega)}(1 + \|\tau(u)\|_{L^p(\Omega)}), 1\} \\ &\leq C\|\tau(u)\|_{L^p(\Omega)}. \end{aligned}$$

Thus, the lemma is true for either $h_0 = h_1$ or $h_0 = 0$. \square

Lemma 4.3. *There exist $\epsilon_0 > 0$ so that if*

$$E(u, D_1 \setminus D_r) \leq \epsilon_0 \quad \text{for some } 0 < r \leq \frac{1}{4}, \quad \|\tau(u)\|_{L^p(D_1)} \leq 1,$$

and there exists a holomorphic function $h_{0,2r}$ in D_{2r} satisfying

$$\|h - h_{0,2r}\|_{L^1(D_{2r})} + r^{\frac{2p-2}{p}} \|\tau(u)\|_{L^p(D_1)} \leq A_0.$$

Then there exists a holomorphic function h_0 in D_1 such that

$$\|h - h_0\|_{L^1(D_1)} \leq C \left(A_0 \ln \frac{1}{r} + \min \left\{ \frac{A_0}{r}, A_0^{\frac{1}{2}} + r^{\frac{1}{2}} \right\} + \|\tau(u)\|_{L^p(D_1)} \right).$$

Here C is a constant independent of A_0 and r .

Proof. Using the same notations as in Lemma 4.2 with $\Omega = D_1 \setminus D_r$, we have

$$(4.1) \quad \|h - h_1\|_{L^1(D_1 \setminus D_r)} \leq C \|h - h_1\|_{L^p(D_1 \setminus D_r)} \leq C \|\tau(u)\|_{L^p(D_1)},$$

We denote by $\sum_{n \in \mathbb{Z}} a_n z^n$ ($a_n \in \mathbb{C}^k$) the Laurent expansion of $G_1(z)$ in $D_1 \setminus D_r$. Then we have

$$h_1(z) = \sum_{n \in \mathbb{Z}} b_n z^n \quad \text{with} \quad b_n = \sum_{m \in \mathbb{Z}} \langle a_m, a_{n-m} \rangle,$$

and we define

$$h_0(z) = \sum_{n=0}^{\infty} b_n z^n.$$

Hence, $h_0(z)$ is holomorphic in D_1 . Since $h_{0,2r}$ is holomorphic in D_{2r} , we may write

$$h_{0,2r}(z) = \sum_{n=0}^{\infty} b'_n z^n \quad \text{in } D_{2r}.$$

Let $r_1 = \frac{5}{4}r$, $r_2 = \frac{3}{2}r$, $r_3 = \frac{7}{4}r$. Then we obtain

$$\begin{aligned} \|h - h_0\|_{L^1(D_1)} &\leq \|h - h_0\|_{L^1(D_1 \setminus D_{r_2})} + \|h - h_0\|_{L^1(D_{r_2})} \\ &\leq \|h - h_1\|_{L^1(D_1 \setminus D_{r_2})} + \|h - h_{0,2r}\|_{L^1(D_{r_2})} \\ &\quad + \|h_1 - h_0\|_{L^1(D_1 \setminus D_{r_2})} + \|h_0 - h_{0,2r}\|_{L^1(D_{r_2})} \\ &\leq \|h - h_1\|_{L^1(D_1 \setminus D_r)} + \|h - h_{0,2r}\|_{L^1(D_{2r})} \\ &\quad + \sum_{n=1}^{\infty} |b_{-n}| \|z^{-n}\|_{L^1(D_1 \setminus D_{r_2})} + \sum_{n=0}^{\infty} |b_n - b'_n| \|z^n\|_{L^1(D_{r_2})} \\ &\leq \|h - h_1\|_{L^1(D_1 \setminus D_r)} + \|h - h_{0,2r}\|_{L^1(D_{2r})} \\ &\quad + C \left(|b_{-1}| + |b_{-2}| \ln \frac{1}{r_2} + \sum_{n=3}^{\infty} |b_{-n}| \frac{1}{r_2^{n-2}} + \sum_{n=0}^{\infty} |b_n - b'_n| r_2^{n+2} \right). \end{aligned}$$

Estimate of b_{-n} .

Thanks to the assumption, we get

$$\begin{aligned} \|h_{0,2r} - h_1\|_{L^1(D_{2r} \setminus D_r)} &\leq \|h - h_{0,2r}\|_{L^1(D_{2r} \setminus D_r)} + Cr^{\frac{2p-2}{p}} \|h - h_1\|_{L^p(D_{2r} \setminus D_r)} \\ &\leq CA_0, \end{aligned}$$

which implies that for $j = 1, 2, 3$,

$$\int_{|z|=r_j} |z| |h_{0,2r} - h_1| |dz| \leq C \|h_{0,2r} - h_1\|_{L^1(D_{2r} \setminus D_r)} \leq CA_0.$$

Hence, we deduce that for $n \geq 1$,

$$|b_{-n}| = \frac{1}{2\pi} \left| \int_{|z|=r_1} z^{n-1} h_1 dz \right| \leq r_1^{n-2} \int_{|z|=r_1} |z| |h_{0,2r} - h_1| |dz| \leq Cr_1^{n-2} A_0,$$

and for $n \geq 0$,

$$|b_n - b'_n| = \frac{1}{2\pi} \left| \int_{|z|=r_3} z^{-n-1} h_1 dz \right| \leq r_3^{-n-2} \int_{|z|=r_3} |z| |h_{0,2r} - h_1| |dz| \leq Cr_3^{-n-2} A_0.$$

Refined estimate of b_{-1} .

Recalling $b_n = \sum_{m \in \mathbb{Z}} \langle a_m, a_{n-m} \rangle$, we get

$$(4.2) \quad |b_{-1}| \leq 2 \left(|a_{-1}| |a_0| + \sum_{n=1}^{\infty} |a_n| |a_{-1-n}| \right),$$

$$(4.3) \quad |\langle a_{-1}, a_{-1} \rangle| \leq |b_{-2}| + 2 \sum_{n=0}^{\infty} |a_n| |a_{-2-n}|.$$

A direct calculation yields that For $r \leq \rho_2 < \rho_1 \leq 1$, we have

$$\begin{aligned} \|z^n\|_{L^2(D_{\rho_1} \setminus D_{\rho_2})}^2 &= \pi \frac{\rho_1^{2n+2} - \rho_2^{2n+2}}{n+1} \quad \text{for } n \neq -1, \\ \|z^{-1}\|_{L^2(D_{\rho_1} \setminus D_{\rho_2})}^2 &= 2\pi \ln \frac{\rho_1}{\rho_2}, \\ \|z^n\|_{L^2(D_{\rho_1} \setminus D_{\rho_2})}^2 &\leq \rho_1^2 \|z^n\|_{L^2(D_1 \setminus D_r)}^2 \quad \text{for } n \geq 0, \\ \|z^n\|_{L^2(D_{\rho_1} \setminus D_{\rho_2})}^2 &\leq (r/\rho_2)^2 \|z^n\|_{L^2(D_1 \setminus D_r)}^2 \quad \text{for } n \leq -2. \end{aligned}$$

Then, using the fact that

$$\|G_1\|_{L^2(D_{\rho_1} \setminus D_{\rho_2})}^2 = \sum_{n \in \mathbb{Z}} |a_n|^2 \|z^n\|_{L^2(D_{\rho_1} \setminus D_{\rho_2})}^2,$$

we infer that

$$(4.4) \quad \|G_1\|_{L^2(D_{\rho_1} \setminus D_{\rho_2})}^2 \leq 2\pi |a_{-1}|^2 \ln \frac{\rho_1}{\rho_2} + \max \left(\rho_1, \frac{r}{\rho_2} \right)^2 \|G_1\|_{L^2(D_1 \setminus D_r)}^2.$$

Using the fact that $\ln \|z^n\|_{L^2(D_1 \setminus D_r)}^2$ is a convex function of n , we obtain

$$\begin{aligned} \|z^n\|_{L^2(D_1 \setminus D_r)} \|z^{-1-n}\|_{L^2(D_1 \setminus D_r)} &\geq \|z^1\|_{L^2(D_1 \setminus D_r)} \|z^{-2}\|_{L^2(D_1 \setminus D_r)} \\ &= \pi \sqrt{\frac{(1-r^4)(1-r^2)}{2r^2}} \end{aligned}$$

for $n \geq 1$, and for $n \geq 0$,

$$\begin{aligned} \|z^n\|_{L^2(D_1 \setminus D_r)} \|z^{-2-n}\|_{L^2(D_1 \setminus D_r)} &\geq \|z^0\|_{L^2(D_1 \setminus D_r)} \|z^{-2}\|_{L^2(D_1 \setminus D_r)} \\ &= \pi \frac{1-r^2}{r}. \end{aligned}$$

Then we conclude that

$$\begin{aligned} \|G_1\|_{L^2(D_1 \setminus D_r)}^2 &= \sum_{n \in \mathbb{Z}} |a_n|^2 \|z^n\|_{L^2(D_1 \setminus D_r)}^2 \\ &\geq |a_0|^2 \|z^0\|_{L^2(D_1 \setminus D_r)}^2 + 2 \sum_{n=1}^{\infty} |a_n| |a_{-1-n}| \|z^n\|_{L^2(D_1 \setminus D_r)} \|z^{-1-n}\|_{L^2(D_1 \setminus D_r)} \\ &\geq |a_0|^2 \pi (1-r^2) + 2 \sum_{n=1}^{\infty} |a_n| |a_{-1-n}| \pi \sqrt{\frac{(1-r^4)(1-r^2)}{2r^2}} \\ &\geq |a_0|^2 + \frac{2}{r} \sum_{n=1}^{\infty} |a_n| |a_{-1-n}|, \end{aligned}$$

and

$$\begin{aligned} \|G_1\|_{L^2(D_1 \setminus D_r)}^2 &= \sum_{n \in \mathbb{Z}} |a_n|^2 \|z^n\|_{L^2(D_1 \setminus D_r)}^2 \\ &\geq 2 \sum_{n=0}^{\infty} |a_n| |a_{-2-n}| \|z^n\|_{L^2(D_1 \setminus D_r)} \|z^{-2-n}\|_{L^2(D_1 \setminus D_r)} \\ &\geq 2 \sum_{n=1}^{\infty} |a_n| |a_{-2-n}| \pi \frac{1-r^2}{r} \\ &\geq \frac{2}{r} \sum_{n=1}^{\infty} |a_n| |a_{-2-n}|, \end{aligned}$$

which along with (4.2) and (4.3) yield that

$$(4.5) \quad |b_{-1}| \leq 2|a_{-1}| \|G_1\|_{L^2(D_1 \setminus D_r)} + r \|G_1\|_{L^2(D_1 \setminus D_r)}^2,$$

$$(4.6) \quad |\langle a_{-1}, a_{-1} \rangle| \leq |b_{-2}| + r \|G_1\|_{L^2(D_1 \setminus D_r)}^2.$$

It remains to estimate a_{-1} . For this, we denote

$$q(z) = q(|z|) = \frac{1}{2\pi} \int_0^{2\pi} R(ze^{i\theta}) d\theta, \quad u = \begin{pmatrix} u \\ 0 \end{pmatrix} \in \mathbb{R}^{2k}.$$

Let $\Omega = D_1 \setminus D_r$. We have

$$\begin{aligned} \left\| \frac{R-q}{\bar{z}} \right\|_{L^2(\Omega)}^2 &= \int_r^1 \int_0^{2\pi} \frac{|R(te^{i\theta}) - q(t)|^2}{t^2} t dt d\theta \\ &\leq \int_r^1 \int_0^{2\pi} \frac{|\partial_\theta R(te^{i\theta})|^2}{t^2} t dt d\theta \\ &\leq \|\nabla R\|_{L^2(\Omega)}^2 \leq C\epsilon_0. \end{aligned}$$

Moreover, we also have

$$A = R^T \frac{\partial u}{\partial z}, \quad G = B^T A = B^T R^T \frac{\partial u}{\partial z},$$

which gives

$$\begin{aligned} G_1 &= G - G_2 = B^T R^T \frac{\partial u}{\partial z} - G_2 \\ &= q^T \frac{\partial u}{\partial z} + (R - q)^T \frac{\partial u}{\partial z} + (B - I_k)^T R^T \frac{\partial u}{\partial z} - G_2. \end{aligned}$$

Therefore for $\rho = \sqrt{r/2}$, we have

$$(2\pi \ln 2)a_{-1} = \int_{D_{2\rho} \setminus D_\rho} \frac{G_1(z)}{\bar{z}} dz \triangleq I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \int_{D_{2\rho} \setminus D_\rho} \frac{q^T}{\bar{z}} \frac{\partial u}{\partial z} dz, \\ I_2 &= \int_{D_{2\rho} \setminus D_\rho} \left(\frac{(R - q)^T}{\bar{z}} \frac{\partial u}{\partial z} + (B - I_k)^T \frac{R^T}{\bar{z}} \frac{\partial u}{\partial z} - \frac{G_2}{\bar{z}} \right) dz. \end{aligned}$$

Notice that

$$\begin{aligned} I_1 &= \int_\rho^{2\rho} \int_0^{2\pi} \frac{q(t)^T}{te^{-i\theta}} \frac{e^{-i\theta}}{2} (\partial_t - i\frac{1}{t}\partial_\theta) u(te^{i\theta}) t dt d\theta \\ &= \int_\rho^{2\rho} \int_0^{2\pi} \frac{q(t)^T}{2} \partial_t u(te^{i\theta}) dt d\theta \in \mathbb{R}^k. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} (2\pi \ln 2)|\operatorname{Im} a_{-1}| &\leq |I_2| \leq \left\| \frac{R - q}{\bar{z}} \right\|_{L^2(\Omega)} \left\| \frac{\partial u}{\partial z} \right\|_{L^2(D_{2\rho} \setminus D_\rho)} + \left\| \frac{1}{\bar{z}} \right\|_{L^2(D_{2\rho} \setminus D_\rho)} \|G_2\|_{L^2(D_{2\rho} \setminus D_\rho)} \\ &\quad + \|B - I_k\|_{L^\infty(\Omega)} \left\| \frac{1}{\bar{z}} \right\|_{L^2(D_{2\rho} \setminus D_\rho)} \left\| \frac{\partial u}{\partial z} \right\|_{L^2(D_{2\rho} \setminus D_\rho)} \\ &\leq C\epsilon_0^{\frac{1}{2}} \left\| \frac{\partial u}{\partial z} \right\|_{L^2(D_{2\rho} \setminus D_\rho)} + C\epsilon_0 \left\| \frac{\partial u}{\partial z} \right\|_{L^2(D_{2\rho} \setminus D_\rho)} + C \|G_2\|_{L^2(D_{2\rho} \setminus D_\rho)} \\ &\leq C\epsilon_0^{\frac{1}{2}} \|G\|_{L^2(D_{2\rho} \setminus D_\rho)} + C \|G_2\|_{L^2(D_{2\rho} \setminus D_\rho)} \\ &\leq C\epsilon_0^{\frac{1}{2}} \|G_1\|_{L^2(D_{2\rho} \setminus D_\rho)} + C \|G_2\|_{L^2(D_{2\rho} \setminus D_\rho)} \\ &\leq C\epsilon_0^{\frac{1}{2}} \left(2\pi \ln 2 |a_{-1}|^2 + (2\rho)^2 \|G_1\|_{L^2(D_1 \setminus D_r)}^2 \right)^{\frac{1}{2}} + C \|G_2\|_{L^2(D_{2\rho} \setminus D_\rho)}, \end{aligned}$$

where we used (4.4) in the last inequality. This along with (4.6) gives

$$\begin{aligned} |a_{-1}|^2 &= \operatorname{Re} \langle a_{-1}, a_{-1} \rangle + 2|\operatorname{Im} a_{-1}|^2 \\ &\leq |b_{-2}| + r \|G_1\|_{L^2(D_1 \setminus D_r)}^2 + C\epsilon_0 \left(|a_{-1}|^2 + r \|G_1\|_{L^2(D_1 \setminus D_r)}^2 \right) + C \|G_2\|_{L^2(D_{2\rho} \setminus D_\rho)}^2 \\ &\leq |b_{-2}| + Cr \|G_1\|_{L^2(D_1 \setminus D_r)}^2 + C \|G_2\|_{L^2(D_{2\rho} \setminus D_\rho)}^2 + C\epsilon_0 |a_{-1}|^2. \end{aligned}$$

Taking ϵ_0 small such that $C\epsilon_0 \leq \frac{1}{2}$, we obtain

$$|a_{-1}|^2 \leq C(|b_{-2}| + r \|G_1\|_{L^2(D_1 \setminus D_r)}^2 + \|G_2\|_{L^2(D_{2\rho} \setminus D_\rho)}^2).$$

Using the facts that

$$\begin{aligned}\|G_1\|_{L^2(D_1 \setminus D_r)} &\leq \|G\|_{L^2(D_1 \setminus D_r)} + \|G_2\|_{L^2(D_1 \setminus D_r)} \\ &\leq C\|\nabla u\|_{L^2(D_1 \setminus D_r)} + C\|\tau\|_{L^p(D_1 \setminus D_r)} \leq C,\end{aligned}$$

and $|b_{-2}| \leq CA_0$ and

$$\begin{aligned}\|G_2\|_{L^2(D_{2\rho} \setminus D_\rho)}^2 &\leq C\rho^{\frac{4p-4}{p}} \|G_2\|_{L^{\frac{2p}{2-p}}(D_1 \setminus D_r)}^2 \leq C\rho^{\frac{4p-4}{p}} \|\tau\|_{L^p(D_1 \setminus D_r)}^2 \\ &\leq Cr^{\frac{2p-2}{p}} \|\tau\|_{L^p(D_1 \setminus D_r)} \leq CA_0,\end{aligned}$$

we deduce that

$$(4.7) \quad |a_{-1}|^2 \leq C(A_0 + r),$$

$$(4.8) \quad |b_{-1}| \leq C(|a_{-1}| + r) \leq C(A_0^{\frac{1}{2}} + r^{\frac{1}{2}}).$$

On the other hand, we also have $|b_{-1}| \leq C\frac{A_0}{r}$, hence,

$$|b_{-1}| \leq C \min\left(\frac{A_0}{r}, A_0^{\frac{1}{2}} + r^{\frac{1}{2}}\right).$$

Collecting the estimates of b_n , we finally conclude that

$$\begin{aligned}\|h - h_0\|_{L^1(D_1)} &\leq C\left(\|\tau(u)\|_{L^p(D_1)} + A_0 \ln \frac{1}{r} + |b_{-1}|\right) \\ &\leq C\left(\|\tau(u)\|_{L^p(D_1)} + A_0 \ln \frac{1}{r} + \min\left(\frac{A_0}{r}, A_0^{\frac{1}{2}} + r^{\frac{1}{2}}\right)\right).\end{aligned}$$

This completes the proof of Lemma 4.3. \square

Lemma 4.4. *Let $0 < r < \rho$. If $h \in L^1(D(z_0, 2r + \rho))$ and for any $z \in D(z_0, r + \rho)$, there exists a holomorphic function $h_{z,r}$ in $D(z, r)$ so that*

$$\|h - h_{z,r}\|_{L^1(D(z,r))} \leq A_0,$$

then there exists a holomorphic function h_0 in $D(z_0, \rho)$ so that

$$\|h - h_0\|_{L^1(D(z_0, \rho))} \leq C\frac{\rho^3}{r^3} A_0.$$

Here C is a constant independent of r, ρ , and A_0 .

Proof. Let ϕ be a radial cut-off function satisfying

$$\text{supp } \phi \subseteq D_{\frac{1}{2}}, \quad \int_{\mathbb{R}^2} \phi dx = 1.$$

Let $\phi_{(r)}(x) = r^{-2}\phi\left(\frac{x}{r}\right)$ and

$$h_{(r)}(z) = \phi_{(r)} * h(z) = \int_{D(z_0, 2r+\rho)} \phi_{(r)}(z-y)h(y)dy$$

for $z \in D(z_0, r + \rho)$. We have $\phi_{(r)} * h_{z,r} = h_{z,r}$ in $D(z, \frac{r}{2})$ due to the mean value equality of holomorphic function. Therefore,

$$\begin{aligned}\|h - h_{(r)}\|_{L^1(D(z, \frac{r}{2}))} &\leq \|h - h_{z,r}\|_{L^1(D(z, \frac{r}{2}))} + \|h_{(r)} - h_{z,r}\|_{L^1(D(z, \frac{r}{2}))} \\ &= \|h - h_{z,r}\|_{L^1(D(z, \frac{r}{2}))} + \|\phi_{(r)} * (h - h_{z,r})\|_{L^1(D(z, \frac{r}{2}))} \\ &\leq \|h - h_{z,r}\|_{L^1(D(z, r))} + \|\phi_{(r)}\|_{L^1} \|h - h_{z,r}\|_{L^1(D(z, r))} \\ &= 2A_0.\end{aligned}$$

Using Fubini theorem, we get

$$\begin{aligned}
\int_{D(z_0, \frac{r}{2} + \rho)} \|h - h_{(r)}\|_{L^1(D(z, \frac{r}{2}))} dz &= \int_{D(z_0, \frac{r}{2})} \|h - h_{(r)}\|_{L^1(D(z, \frac{r}{2} + \rho))} dz \\
&\geq \int_{D(z_0, \frac{r}{2})} \|h - h_{(r)}\|_{L^1(D(z_0, \rho))} dz \\
&= \pi \left(\frac{r}{2}\right)^2 \|h - h_{(r)}\|_{L^1(D(z_0, \rho))},
\end{aligned}$$

which implies

$$\begin{aligned}
\|h - h_{(r)}\|_{L^1(D(z_0, \rho))} &\leq \left(1 + \frac{2\rho}{r}\right)^2 \sup_{z \in D(z_0, \frac{r}{2} + \rho)} \|h - h_{(r)}\|_{L^1(D(z, \frac{r}{2}))} \\
(4.9) \quad &\leq \left(1 + \frac{2\rho}{r}\right)^2 (2A_0).
\end{aligned}$$

Notice that in $D(z, r/2)$, we have

$$\begin{aligned}
\frac{\partial}{\partial \bar{z}} h_{(r)} &= \frac{\partial}{\partial \bar{z}} (h_{(r)} - h_{z,r}) = \frac{\partial}{\partial \bar{z}} (\phi_{(r)} * (h - h_{z,r})) \\
&= \left(\frac{\partial}{\partial \bar{z}} \phi_{(r)}\right) * (h - h_{z,r}).
\end{aligned}$$

Therefore,

$$\left\| \frac{\partial}{\partial \bar{z}} h_{(r)} \right\|_{L^1(D(z, \frac{r}{2}))} \leq \left\| \frac{\partial}{\partial \bar{z}} \phi_{(r)} \right\|_{L^1} \|h - h_{z,r}\|_{L^1(D(z, r))} \leq \frac{C}{r} A_0,$$

which implies (similar to (4.9))

$$\begin{aligned}
\left\| \frac{\partial}{\partial \bar{z}} h_{(r)} \right\|_{L^1(D(z_0, \rho))} &\leq \left(1 + \frac{2\rho}{r}\right)^2 \sup_{z \in D(z_0, \frac{r}{2} + \rho)} \left\| \frac{\partial}{\partial \bar{z}} h_{(r)} \right\|_{L^1(D(z, \frac{r}{2}))} \\
&\leq \left(1 + \frac{2\rho}{r}\right)^2 \frac{C}{r} A_0.
\end{aligned}$$

Now we let $h_1 = (\frac{1}{\pi z}) * (\frac{\partial}{\partial \bar{z}} h_{(r)} \chi_{D(z_0, \rho)})$ and $h_0 = h_{(r)} - h_1$. Then h_0 is holomorphic in $D(z_0, \rho)$ and by (4.9),

$$\begin{aligned}
\|h - h_0\|_{L^1(D(z_0, \rho))} &= \|h - h_{(r)} + h_1\|_{L^1(D(z_0, \rho))} \\
&\leq \|h - h_{(r)}\|_{L^1(D(z_0, \rho))} + \|h_1\|_{L^1(D(z_0, \rho))} \\
&\leq \left(1 + \frac{2\rho}{r}\right)^2 \left(2A_0 + \frac{C\rho}{r} A_0\right) \\
&\leq C \frac{\rho^3}{r^3} A_0,
\end{aligned}$$

here we used

$$\|h_1\|_{L^1(D(z_0, \rho))} \leq \left\| \frac{1}{\pi z} \right\|_{L^1(D_{2\rho})} \left\| \frac{\partial}{\partial \bar{z}} h_{(r)} \right\|_{L^1(D(z_0, \rho))} \leq \left(1 + \frac{2\rho}{r}\right)^2 \frac{C\rho}{r} A_0.$$

This completes the proof of Lemma 4.4. □

Now we are in a position to prove Proposition 4.1.

Proof of Proposition 4.1. We use the induction argument. The case of $m = 1$ follows from Lemma 4.2. Let us assume that the case of $m - 1$ is true. The proof of the assertion for m is split it into many cases.

Case 1. $\|\tau(u)\|_{L^p(D_1)} \geq 1$.

In this case, we may take $h_0 = 0$, since

$$\|h\|_{L^1(D_1)} \leq \left\| \frac{\partial u}{\partial z} \right\|_{L^2(D_1)}^2 \leq E(u, D_1) \leq m\epsilon_0 \leq m\epsilon_0 \|\tau(u)\|_{L^p(D_1)}^{3^{1-m}}.$$

Case 2. $\|\tau(u)\|_{L^p(D_1)} \leq 1$ and $E(u, D_{\frac{1}{4}}) \leq \epsilon_0$.

We first consider the function $v(z) = u(\frac{z}{4})$, which satisfies

$$E(v, D_1) = E(u, D_{\frac{1}{4}}) \leq \epsilon_0, \quad \tau(v)(z) = \frac{1}{16}\tau(u)\left(\frac{z}{4}\right),$$

hence,

$$\|\tau(v)\|_{L^p(D_1)} = \left(\frac{1}{4}\right)^{\frac{2p-2}{p}} \|\tau(u)\|_{L^p(D_{\frac{1}{4}})} \leq \|\tau(u)\|_{L^p(D_1)} \leq 1.$$

Then Lemma 4.2 ensures that there exists a holomorphic function \tilde{h}_0 in D_1 so that

$$\|\tilde{h} - \tilde{h}_0\|_{L^1(D_1)} \leq C\|\tau(v)\|_{L^p(D_1)},$$

where $\tilde{h}(z) = \langle \frac{\partial v}{\partial z}, \frac{\partial v}{\partial \bar{z}} \rangle = \frac{1}{16}h(\frac{z}{4})$. Therefore, $h_0(z) = 16\tilde{h}_0(4z)$ is holomorphic in $D_{\frac{1}{4}}$ and satisfies

$$\begin{aligned} \|h - h_0\|_{L^1(D_{\frac{1}{4}})} &= \|\tilde{h} - \tilde{h}_0\|_{L^1(D_1)} \\ &\leq C\|\tau(v)\|_{L^p(D_1)} \leq C\|\tau(u)\|_{L^p(D_1)} \leq C\|\tau(u)\|_{L^p(D_1)}^{3^{1-m}}. \end{aligned}$$

Case 3. $\|\tau(u)\|_{L^p(D_1)} \leq 1$ and $Q(\frac{1}{8}) \leq (m-1)\epsilon_0$, where

$$Q(t) \stackrel{\text{def}}{=} \sup_{D(z,t) \subseteq D_1} E(u, D(z,t)) \quad \text{for } t \in [0, 1].$$

Obvious, $Q(t)$ is continuous and non-decreasing in t and $Q(0) = 0$. In this case, we let $v(z) = u(z' + \frac{z}{8})$ for $z' \in D_{1/2}$. Then v satisfies

$$E(v, D_1) = E(u, D(z', \frac{1}{8})) \leq Q(\frac{1}{8}) \leq (m-1)\epsilon_0,$$

and $\tau(v)(z) = \frac{1}{64}\tau(u)(z' + \frac{z}{8})$, hence,

$$\|\tau(v)\|_{L^p(D_1)} = \left(\frac{1}{8}\right)^{\frac{2p-2}{p}} \|\tau(u)\|_{L^p(D(z', \frac{1}{8}))} \leq \|\tau(u)\|_{L^p(D_1)}.$$

Then the induction assumption ensures that there exists a holomorphic function \tilde{h}_0 in $D_{\frac{1}{4}}$ so that

$$\|\tilde{h} - \tilde{h}_0\|_{L^1(D_{\frac{1}{4}})} \leq C_{m-1}\|\tau(v)\|_{L^p(D_1)}^{3^{2-m}}.$$

where $\tilde{h} = \langle \frac{\partial v}{\partial z}, \frac{\partial v}{\partial \bar{z}} \rangle = \frac{1}{64}h(z' + \frac{z}{8})$. Set $r = \frac{1}{32}$. Then $h_{z',r}(z) = 64\tilde{h}_0(8(z - z'))$ is holomorphic in $D(z', \frac{1}{32})$ and satisfies

$$\|h - h_{z',r}\|_{L^1(D(z', \frac{1}{32}))} = \|\tilde{h} - \tilde{h}_0\|_{L^1(D_{\frac{1}{4}})} \leq C_{m-1}\|\tau(v)\|_{L^p(D_1)}^{3^{2-m}} \leq C_{m-1}\|\tau(u)\|_{L^p(D_1)}^{3^{2-m}}.$$

Therefore, h satisfies the conditions in Lemma 4.4 with $z_0 = 0, r = \frac{1}{32}, \rho = \frac{1}{4}$ and $A_0 = C_{m-1} \|\tau(u)\|_{L^p(D_1)}^{3^{2-m}}$. Thus, Lemma 4.4 ensures the existence of a holomorphic function h_0 in $D_{\frac{1}{4}}$ so that

$$\|h - h_0\|_{L^1(D_{\frac{1}{4}})} \leq C \frac{\rho^3}{r^3} A_0 = 8^3 C C_{m-1} \|\tau(u)\|_{L^p(D_1)}^{3^{2-m}} \leq C C_{m-1} \|\tau(u)\|_{L^p(D_1)}^{3^{1-m}}.$$

Case 4. $\|\tau(u)\|_{L^p(D_1)} \leq 1$, $Q(\frac{1}{8}) > (m-1)\epsilon_0$ and $E(u, D_{\frac{1}{4}}) > \epsilon_0$.

In this case, there exists $0 < r_0 < \frac{1}{8}$ such that $Q(r_0) = (m-1)\epsilon_0$. Thus, there exists $D(z_0, r_0) \subseteq D_1$ so that $E(u, D(z_0, r_0)) = (m-1)\epsilon_0$. Hence,

$$E(u, D(z_0, r_0)) + E(u, D_{\frac{1}{4}}) > E(u, D_1),$$

which implies that $D(z_0, r_0) \cap D_{\frac{1}{4}} \neq \emptyset$, thus $|z_0| \leq \frac{1}{4} + r_0$ and then $D(z_0, 4r_0) \subseteq D_1$. For $z' \in D(z_0, 3r_0)$, the function $v(z) = u(z' + r_0 z)$ satisfies

$$E(v, D_1) = E(u, D(z', r_0)) \leq Q(r_0) = (m-1)\epsilon_0,$$

and $\tau(v)(z) = r_0^2 \tau(u)(z' + r_0 z)$, hence,

$$\|\tau(v)\|_{L^p(D_1)} = r_0^{\frac{2p-2}{p}} \|\tau(u)\|_{L^p(D(z', r_0))} \leq r_0^{\frac{2p-2}{p}} \|\tau(u)\|_{L^p(D_1)}.$$

Then following argument of Case 3 (using Lemma 4.4 with $A_0 = C_{m-1} \left(r_0^{\frac{2p-2}{p}} \|\tau(u)\|_{L^p(D_1)} \right)^{3^{2-m}}$, $r = \frac{r_0}{4}, \rho = 2r_0$), we can conclude the existence of a holomorphic function \tilde{h}_0 in $D(z_0, 2r_0)$ such that

$$\|h - \tilde{h}_0\|_{L^1(D(z_0, 2r_0))} \leq C \frac{\rho^3}{r^3} A_0 = 8^3 C C_{m-1} \left(r_0^{\frac{2p-2}{p}} \|\tau(u)\|_{L^p(D_1)} \right)^{3^{2-m}}.$$

Now set $\rho_0 = \frac{1}{2} + r_0$, $r = r_0/\rho_0$ and consider the function $v(z) = u(\rho_0 z + z_0)$. Then we have

$$E(v, D_1 \setminus D_r) = E(u, D(z_0, \rho_0) \setminus D(z_0, r_0)) \leq E(u, D_1) - E(u, D(z_0, r_0)) \leq \epsilon_0,$$

$$\|\tau(v)\|_{L^p(D_1)} = \rho_0^{\frac{2p-2}{p}} \|\tau(u)\|_{L^p(D(z', \rho_0))} \leq \|\tau(u)\|_{L^p(D_1)} \leq 1.$$

For $0 < r_0 < r < \frac{1}{4}$, the function $h_{0,2r}(z) = \rho_0^2 \tilde{h}_0(\rho_0 z + z_0)$ is holomorphic in D_{2r} and satisfies

$$\begin{aligned} \|\tilde{h} - h_{0,2r}\|_{L^1(D_{2r})} + r^{\frac{2p-2}{p}} \|\tau(v)\|_{L^p(D_1)} &= \|h - \tilde{h}_0\|_{L^1(D(z_0, 2r_0))} + r^{\frac{2p-2}{p}} \|\tau(v)\|_{L^p(D_1)} \\ &\leq 8^3 C C_{m-1} \left(r_0^{\frac{2p-2}{p}} \|\tau(u)\|_{L^p(D_1)} \right)^{3^{2-m}} + r^{\frac{2p-2}{p}} \|\tau(u)\|_{L^p(D_1)} \\ &\leq C C_{m-1} \left(r^{\frac{2p-2}{p}} \|\tau(u)\|_{L^p(D_1)} \right)^{3^{2-m}}, \end{aligned}$$

here $\tilde{h} = \langle \frac{\partial v}{\partial z}, \frac{\partial v}{\partial \bar{z}} \rangle = \rho_0^2 h(\rho_0 z + z_0)$. Therefore, v and \tilde{h} satisfies the conditions of Lemma 4.3 with

$$(4.10) \quad A_0 = C C_{m-1} \left(r^{\frac{2p-2}{p}} \|\tau(u)\|_{L^p(D_1)} \right)^{3^{2-m}}.$$

Thus, there exists a holomorphic function \hat{h}_0 in D_1 such that

$$\|\tilde{h} - \hat{h}_0\|_{L^1(D_1)} \leq C \left(A_0 \ln \frac{1}{r} + \min \left\{ \frac{A_0}{r}, A_0^{\frac{1}{2}} + r^{\frac{1}{2}} \right\} + \|\tau(u)\|_{L^p(D_1)} \right).$$

For A_0 given by (4.10), we have

$$\begin{aligned} A_0 \ln \frac{1}{r} &\leq C \frac{p}{2p-2} 3^{m-2} C_{m-1} \|\tau(u)\|_{L^p(D_1)}^{3^{2-m}} \leq C 3^m C_{m-1} \|\tau(u)\|_{L^p(D_1)}^{3^{1-m}}, \\ \min \left(\frac{A_0}{r}, A_0^{\frac{1}{2}} + r^{\frac{1}{2}} \right) &\leq A_0^{\frac{1}{2}} + \min \left(\frac{A_0}{r}, r^{\frac{1}{2}} \right) \leq A_0^{\frac{1}{2}} + A_0^{\frac{1}{3}} \leq C C_{m-1} \|\tau(u)\|_{L^p(D_1)}^{3^{1-m}}. \end{aligned}$$

This gives

$$\|\tilde{h} - \hat{h}_0\|_{L^1(D_1)} \leq C 3^m C_{m-1} \|\tau(u)\|_{L^p(D_1)}^{3^{1-m}}.$$

Let $h_0(z) = \rho_0^{-2} \hat{h}_0(\frac{z-z_0}{\rho_0})$. Then $h_0(z)$ is holomorphic in $D(z_0, \rho_0)$ and satisfies

$$\begin{aligned} \|h - h_0\|_{L^1(D_{\frac{1}{4}})} &\leq \|h - h_0\|_{L^1(D(z_0, \rho_0))} \\ &= \|\tilde{h} - \hat{h}_0\|_{L^1(D_1)} \leq C 3^m C_{m-1} \|\tau(u)\|_{L^p(D_1)}^{3^{1-m}}. \end{aligned}$$

Therefore, the assertion holds for m with $C_m = C 3^m C_{m-1}$. The proof of Proposition 4.1 is completed.

5. PROOF OF THEOREM 1.1

Let us first prove the following energy decay estimates for the map satisfying the assumptions of Lemma 4.3.

Lemma 5.1. *Let u be as in Lemma 4.3. Then it holds that*

$$\begin{aligned} E(u, D_\rho \setminus D_{r/\rho}) &\leq C \left((A_0 + r) \ln \frac{\rho^2}{r} + \rho^{\frac{4p-4}{p}} \right) \quad \text{for } \sqrt{r} \leq \rho \leq 1, \\ \text{osc}(u, D_\rho \setminus D_{r/\rho}) &\leq C \left((A_0 + r)^{\frac{1}{2}} \ln \frac{1}{r} + \rho^{\frac{2p-2}{p}} \right) \quad \text{for } 2\sqrt{r} \leq \rho \leq \frac{1}{2}. \end{aligned}$$

Proof. Here we will use the notations in the proof of Lemma 4.3. By (4.4) and (4.7), we get

$$\begin{aligned} E(u, D_\rho \setminus D_{r/\rho}) &\leq C \|G\|_{L^2(D_\rho \setminus D_{r/\rho})}^2 \leq C \left(\|G_1\|_{L^2(D_\rho \setminus D_{r/\rho})}^2 + \|G_2\|_{L^2(D_\rho \setminus D_{r/\rho})}^2 \right) \\ &\leq C \left(|a_{-1}|^2 \ln \frac{\rho^2}{r} + \rho^2 \|G_1\|_{L^2(D_1 \setminus D_r)}^2 + \rho^{\frac{4p-4}{p}} \|G_2\|_{L^{\frac{2p}{2-p}}(D_1 \setminus D_r)}^2 \right) \\ &\leq C \left((A_0 + r) \ln \frac{\rho^2}{r} + \rho^2 + \rho^{\frac{4p-4}{p}} \|\tau\|_{L^p(D_1 \setminus D_r)} \right) \\ &\leq C \left((A_0 + r) \ln \frac{\rho^2}{r} + \rho^{\frac{4p-4}{p}} \right). \end{aligned}$$

If $2\sqrt{r} \leq \rho \leq \frac{1}{2}$, there exists a positive integer $\ell > 1$ such that $e^{-\ell}\rho < r/\rho \leq e^{-\ell+1}\rho$ and $\ell \leq \ln \frac{\rho^2}{r} + 1$. Let $\rho_j = e^{-j}\rho$ for $0 \leq j < \ell$ and $\rho_\ell = r/\rho$. Then by (4.4) and Sobolev

embedding and the fact that G_1 is holomorphic, we deduce that for $0 < j \leq \ell$,

$$\begin{aligned}
\text{osc}(u, D_{\rho_{j-1}} \setminus D_{\rho_j}) &\leq C \rho_j^{\frac{2p-2}{p}} \|\nabla u\|_{L^{\frac{2p}{2-p}}(D_{e\rho_j} \setminus D_{\rho_j})} \leq C \rho_j^{\frac{2p-2}{p}} \|G\|_{L^{\frac{2p}{2-p}}(D_{e\rho_j} \setminus D_{\rho_j})} \\
&\leq C \rho_j^{\frac{2p-2}{p}} \left(\|G_1\|_{L^{\frac{2p}{2-p}}(D_{e\rho_j} \setminus D_{\rho_j})} + \|G_2\|_{L^{\frac{2p}{2-p}}(D_{e\rho_j} \setminus D_{\rho_j})} \right) \\
&\leq C \left(\|G_1\|_{L^2(D_{2e\rho_j} \setminus D_{\rho_j/2})} + \rho_j^{\frac{2p-2}{p}} \|G_2\|_{L^{\frac{2p}{2-p}}(D_1 \setminus D_r)} \right) \\
&\leq C \left(|a_{-1}| + \max\{2e\rho_j, 2r/\rho_j\} \|G_1\|_{L^2(D_1 \setminus D_r)} + \rho_j^{\frac{2p-2}{p}} \|\tau\|_{L^p(D_1 \setminus D_r)} \right) \\
&\leq C \left(|a_{-1}| + \max\{2e\rho_j, 2r/\rho_j\} + \rho_j^{\frac{2p-2}{p}} \right),
\end{aligned}$$

which gives

$$\begin{aligned}
\text{osc}(u, D_\rho \setminus D_{r/\rho}) &\leq \sum_{j=1}^{\ell} \text{osc}(u, D_{\rho_{j-1}} \setminus D_{\rho_j}) \\
&\leq C \sum_{j=1}^{\ell} \left(|a_{-1}| + \max(2e\rho_j, 2r/\rho_j) + \rho_j^{\frac{2p-2}{p}} \right) \\
&\leq C \left(|a_{-1}| \ell + \rho + \rho^{\frac{2p-2}{p}} \right) \\
&\leq C \left((A_0 + r)^{\frac{1}{2}} \ln \frac{1}{r} + \rho^{\frac{2p-2}{p}} \right).
\end{aligned}$$

The proof is finished. \square

Now let us complete the proof of Theorem 1.1. By the construction of bubble tree, it is sufficient to prove (2.1) and (2.2) under the assumption of $E(u_n, D(x_n^j, r_n^{f(j)} \delta_0) \setminus D(x_n^j, r_n^j R_0)) \leq \epsilon$.

We can also assume $\delta_0^{\frac{2p-2}{p}} \Lambda \leq 1$, $r_n^i \leq 1$. Let m be a positive integer so that $m > \Lambda^2/\epsilon$. Let $r = R_0 r_n^i / (\delta_0 r_n^{f(i)})$ and consider the function $w_n^i(z) = u_n(x_n^i + \delta_0 r_n^{f(i)} z)$. It holds that

$$\begin{aligned}
E(w_n^i, D_1 \setminus D_r) &= E(u_n, D(x_n^i, r_n^{f(i)} \delta_0) \setminus D(x_n^i, r_n^i R_0)) \leq \epsilon, \\
\|\tau(w_n^i)\|_{L^p(D_1)} &= (\delta_0 r_n^{f(i)})^{\frac{2p-2}{p}} \|\tau(u_n)\|_{L^p(D(x_n^i, r_n^{f(i)} \delta_0))} \leq \delta_0^{\frac{2p-2}{p}} \Lambda \leq 1.
\end{aligned}$$

For n sufficiently large, $0 < r_n^i < r < \frac{1}{8}$. We consider the function $v_n^i(z) = w_n^i(8rz) = u_n(x_n^i + 8R_0 r_n^i z)$, which satisfies

$$\begin{aligned}
E(v_n^i, D_1) &= E(u_n, D(x_n^i, 8r_n^i R_0)) \leq \Lambda^2 \leq m\epsilon, \\
\|\tau(v_n^i)\|_{L^p(D_1)} &= (8r)^{\frac{2p-2}{p}} \|\tau(w_n^i)\|_{L^p(D_{8r})} \leq (8r)^{\frac{2p-2}{p}} \leq 1.
\end{aligned}$$

Thus, Proposition 4.1 and scaling argument ensure that there exists a holomorphic function $h_{0,2r}^{n,i}$ in D_{2r} such that

$$\|h^{n,i} - h_{0,2r}^{n,i}\|_{L^1(D_{2r})} \leq C_m \|\tau(v_n^i)\|_{L^p(D_1)}^{3^{1-m}} \leq C_m (8r)^{\frac{2p-2}{p} 3^{1-m}}.$$

Hence, we get

$$\begin{aligned} \|h^{n,i} - h_{0,2r}^{n,i}\|_{L^1(D_{2r})} + r^{\frac{2p-2}{p}} \|\tau(w_n^i)\|_{L^p(D_1)} &= C_m (8r)^{\frac{2p-2}{p} 3^{1-m}} + r^{\frac{2p-2}{p}} \\ &\leq C C_m r^{\frac{2p-2}{p} 3^{1-m}}, \end{aligned}$$

here $h^{n,i} = \langle \frac{\partial w_n^i}{\partial z}, \frac{\partial w_n^i}{\partial \bar{z}} \rangle = (\delta_0 r_n^{f(i)})^2 h_n(x_n^i + \delta_0 r_n^{f(i)} z)$. Thus, the conditions in Lemma 4.3 are satisfied for w_n^i with $A_0 = C C_m r^{\frac{2p-2}{p} 3^{1-m}}$. For this A_0 , it holds that

$$\begin{aligned} \lim_{n \rightarrow \infty} (A_0 + r) \ln \frac{\rho^2}{r} &= \lim_{n \rightarrow \infty} (C C_m r^{\frac{2p-2}{p} 3^{1-m}} + r) \ln \frac{\rho^2}{r} = 0, \\ \lim_{n \rightarrow \infty} (A_0 + r)^{\frac{1}{2}} \ln \frac{1}{r} &= \lim_{n \rightarrow \infty} (C C_m r^{\frac{2p-2}{p} 3^{1-m}} + r)^{\frac{1}{2}} \ln \frac{1}{r} = 0, \end{aligned}$$

as $r_n^i \rightarrow 0$ (recall $r = R_0 r_n^i / (\delta_0 r_n^{f(i)}) \rightarrow 0$). Then we apply Lemma 5.1 to conclude that

$$\begin{aligned} \limsup_{n \rightarrow \infty} E(u_n, D(x_n^i, \rho r_n^{f(i)} \delta_0) \setminus D(x_n^i, r_n^i R_0 / \rho)) &\leq C \rho^{\frac{4p-4}{p}}, \\ \limsup_{n \rightarrow \infty} \text{osc}(u_n, D(x_n^i, \rho r_n^{f(i)} \delta_0) \setminus D(x_n^i, r_n^i R_0 / \rho)) &\leq C \rho^{\frac{2p-2}{p}}, \end{aligned}$$

which yield (2.1) and (2.2) by taking $\rho \rightarrow 0$. The proof of Theorem 1.1 is completed. \square

6. A NECESSARY AND SUFFICIENT CONDITION OF ENERGY IDENTITY

Let us first recall the following result [1, 4].

Lemma 6.1. *If $\tau(u_n)$ is bounded in L^p for some $p > 1$, then the tangential energy on the neck domain is vanishing, i.e.,*

$$\lim_{\delta \rightarrow 0} \lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{D(0, r_n^{f(i)} \delta) \setminus D(0, r_n^i R)} |x|^{-2} |\partial_\theta u_n(x_n^i + x)|^2 dx = 0.$$

Notice that

$$\begin{aligned} h_n(x_n^i + x) &= h_n(x_n^i + r e^{i\theta}) \\ &= \frac{1}{4} e^{-2i\theta} \left(|\partial_r u_n|^2 - r^{-2} |\partial_\theta u_n|^2 - \frac{2i}{r} \langle \partial_r u_n, \partial_\theta u_n \rangle \right) (x_n^i + r e^{i\theta}), \end{aligned}$$

which motivates the following equivalent statement of the energy identity.

Proposition 6.2. *Let $h_n = \langle \frac{\partial u_n}{\partial z}, \frac{\partial u_n}{\partial \bar{z}} \rangle$. Then the energy identity holds if and only if*

$$(6.1) \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \|h_n\|_{L^1(D_\delta)} = 0.$$

Proof. On the one hand, if (6.1) holds, then we have by Lemma 6.1 that

$$\begin{aligned} &\lim_{\delta \rightarrow 0} \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} E(u_n, D(x_n^i, r_n^{f(i)} \delta) \setminus D(x_n^i, r_n^i R)) \\ &= \lim_{\delta \rightarrow 0} \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{2} \int_{D(0, r_n^{f(i)} \delta) \setminus D(0, r_n^i R)} (|\partial_r u_n|^2 - |x|^{-2} |\partial_\theta u_n|^2) (x_n^i + x) dx \\ &\leq \lim_{\delta \rightarrow 0} \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} 2 \int_{D(0, r_n^{f(i)} \delta) \setminus D(0, r_n^i R)} |h_n(x_n^i + x)| dx \\ &= \lim_{\delta \rightarrow 0} \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} 2 \|h_n\|_{L^1(D(x_n^i, r_n^{f(i)} \delta) \setminus D(x_n^i, r_n^i R))} \\ &\leq \lim_{\delta \rightarrow 0} \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} 2 \|h_n\|_{L^1(D_\delta)} = 0, \end{aligned}$$

which gives (2.1), thus the energy identity.

On the other hand, if the energy identity holds, we denote

$$w_n^i(z) = u_n(x_n^i + r_n^i z), \quad h_n^i = \left\langle \frac{\partial w_n^i}{\partial z}, \frac{\partial w_n^i}{\partial z} \right\rangle, \quad h^i = \left\langle \frac{\partial w^i}{\partial z}, \frac{\partial w^i}{\partial z} \right\rangle$$

for $i = 1, \dots, k_0$. Then $h_n^i(z) = (r_n^i)^2 h_n(x_n^i + r_n^i z)$, and h^i is a L^1 holomorphic function in \mathbb{C} , thus $h^i = 0$. Thanks to

$$u_n^i \rightarrow w^i \text{ strongly in } W^{1,2}(D_R \setminus \cup_{x \in Z_i} D(x, \delta)) \text{ as } n \rightarrow \infty,$$

we infer that as $n \rightarrow \infty$,

$$\begin{aligned} h_n^i &\rightarrow h^i = 0 \quad \text{strongly in } L^1(D_R \setminus \cup_{x \in Z_i} D(x, \delta)), \\ \|h_n\|_{L^1(D(x_n^i, r_n^i R) \setminus \cup_{x \in Z_i} D(x_n^i + r_n^i x, r_n^i \delta))} &= \|h_n^i\|_{L^1(D_R \setminus \cup_{x \in Z_i} D(x, \delta))} \rightarrow 0. \end{aligned}$$

Notice that

$$\begin{aligned} \|h_n\|_{L^1(D_\delta)} &\leq \sum_{i=1}^{k_0} \|h_n\|_{L^1(D(x_n^i, r_n^i R) \setminus \cup_{x \in Z_i} D(x_n^i + r_n^i x, r_n^i \delta))} + \sum_{i=1}^{k_0} \|h_n\|_{L^1(D(x_n^i, r_n^{f(i)} \delta) \setminus D(x_n^i, r_n^i R))} \\ &\leq \sum_{i=1}^{k_0} \|h_n\|_{L^1(D(x_n^i, r_n^i R) \setminus \cup_{x \in Z_i} D(x_n^i + r_n^i x, r_n^i \delta))} \\ &\quad + \sum_{i=1}^{k_0} E(u_n, D(x_n^i, r_n^{f(i)} \delta) \setminus D(x_n^i, r_n^i R)). \end{aligned}$$

Thus, (6.1) follows easily from (2.1). \square

Using Lemma 6.2 and Proposition 4.1, let us present another proof of the energy identity.

Let m be a positive integer so that $m > \Lambda^2/\epsilon$. Then for fixed $0 < \delta_1 < 1$, we consider the function $w_n(z) = u_n(\delta_1 z)$, which satisfies

$$\begin{aligned} E(w_n, D_1) &= E(u_n, D_{\delta_1}) \leq \Lambda^2 \leq m\epsilon, \\ \|\tau(w_n)\|_{L^p(D_1)} &= \delta_1^{\frac{2p-2}{p}} \|\tau(u_n)\|_{L^p(D_{\delta_1})} \leq \delta_1^{\frac{2p-2}{p}} \Lambda. \end{aligned}$$

Thus, Proposition 4.1 and scaling argument ensure that there exists a holomorphic function $h_{0,n}$ in $D_{\delta_1/4}$ such that

$$\|h_n - h_{0,n}\|_{L^1(D_{\delta_1/4})} \leq C_m \|\tau(w_n)\|_{L^p(D_1)}^{3^{1-m}} \leq CC_m(\delta_1)^{\frac{2p-2}{p} 3^{1-m}}.$$

Therefore, for $0 < \delta < \delta_1/4$,

$$\begin{aligned} \|h_n\|_{L^1(D_\delta)} &\leq \|h_n - h_{0,n}\|_{L^1(D_\delta)} + \|h_{0,n}\|_{L^1(D_\delta)} \\ &\leq \|h_n - h_{0,n}\|_{L^1(D_{\delta_1/4})} + (4\delta/\delta_1)^2 \|h_{0,n}\|_{L^1(D_{\delta_1/4})} \\ &\leq 2\|h_n - h_{0,n}\|_{L^1(D_{\delta_1/4})} + (4\delta/\delta_1)^2 \|h_n\|_{L^1(D_{\delta_1/4})} \\ &\leq CC_m(\delta_1)^{\frac{2p-2}{p} 3^{1-m}} + (4\delta/\delta_1)^2 \Lambda^2. \end{aligned}$$

Then (6.1) follows by first letting $\delta \rightarrow 0$, then letting $\delta_1 \rightarrow 0$.

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REFERENCES

- [1] W. Ding and G. Tian, *Energy identity for a class of approximate harmonic maps from surfaces*, Comm. Anal. Geom., 3(1995), 543-554.
- [2] F. Hélein, *Harmonic Maps, Conservation Laws and Moving Frames*, Diderot, Paris, 1997.
- [3] J. Li and X. Zhu, *Small energy compactness for approximate harmonic mappings*, Comm. Contemp. Math., 13(2011), 741-763.
- [4] J. Li and X. Zhu, *Energy identity for the maps from a surface with tension field bounded in L^p* , Pacific J. Math., 260 (2012), 181-195.
- [5] F.-H. Lin and C. Wang, *Energy identity of harmonic map flows from surfaces at finite singular time*, Calc. Var. Partial Differential Equations, 6(1998), 369-380.
- [6] F.-H. Lin and C. Wang, *Harmonic and quasi-harmonic spheres, II*, Comm. Anal. Geom., 10 (2002), 341-375.
- [7] F.-H. Lin and C. Wang, *The analysis of harmonic maps and their heat flows*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.
- [8] Y. Luo, *Energy identity and removable singularities of maps from a riemann surface with tension field unbounded in L^2* , Pacific J. Math., 256 (2012), 365-380.
- [9] D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology*, American Mathematical Society Colloquium Publications, 52, American Mathematical Society, Providence, RI, 2012.
- [10] T. H. Parker, *Bubble tree convergence for harmonic maps*, J. Differential Geom., 44 (1996), 595-633.
- [11] J. Qing, *On singularities of the heat flow for harmonic maps from surfaces into spheres*, Comm. Anal. Geom., 3(1995), 297-315.
- [12] J. Qing and G. Tian, *Bubbling of the heat flows for harmonic maps from surfaces*, Comm. Pure Appl. Math., 50(1997), 295-310.
- [13] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Ann. of Math., 113(1981), 1-24.
- [14] P. Topping, *Repulsion and quantization in almost-harmonic maps, and asymptotics of the harmonic map flow*, Ann. of Math., 159(2004), 465-534.
- [15] P. Topping, *Winding behaviour of finite-time singularities of the harmonic map heat flow*, Math. Z., 247 (2004), 279-302.
- [16] C. Wang, *Bubble phenomena of certain Palais-Smale sequences from surfaces to general targets*, Houston J. Math., 22(1996), 559-590.
- [17] X. Zhu, *No neck for approximate harmonic maps to the sphere*, Nonlinear Anal., 75(2012), 4339-4345.
- [18] X. Zhu, *Bubble tree for approximate harmonic maps*, Proc. Amer. Math. Soc., 142 (2014), 2849-2857.

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